$$I = \frac{TW}{6}\left[\sin^2(\phi/4)\{L2^2 + (L1/2)^2\}(W^2 - T^2)/L2^2 + T^2\right] \quad (5)$$

$$\phi = 2\pi/N \quad (6)$$

From the work to date an effective width of flange of 21 times the thickness appears realistic.

The solution of Eq. (4) is multivalued. The lowest root (other than zero) corresponds to the fundamental buckling mode of the diagonal member and provides an estimate of the buckling load in that member. The total load on the cylinder (P) is then obtained from equilibrium conditions [Eq. (7)].

$$P = 2NPI\left\{\frac{L2^2 - (L1/2)^2\tan^2(\phi/4)}{L2^2 + (L1/2)^2}\right\} \quad (7)$$

The load calculated from Eq. (7) using values of N given by Eq. (3) is plotted in Fig. 2 as a function of radius to thickness ratio. Since the discussion in this Note is centered around high quality shells where naturally occurring defects are very small we would expect that the calculated buckling loads would not be less than experimentally determined loads. Therefore, the theoretical curve should approximate the upper limit of experimental data. Figure 2 illustrates good correlation between the calculated values and the known experimental evidence for R/T values greater than 1000. The experimental loads never exceed the calculated values by more than 8% and except for radius to thickness ratios in the vicinity of 4000 the agreement is within about 3%.

Discussion

The space frame analogy was developed on the assumption that the ratio of effective width of flange of frame members to shell thickness remained constant. This clearly could not be the case in very small facets where the flanges could be considered larger than the facets. Facets of the size given by Eq. (2) are about the size where the flanges almost completely cover the entire facet (i.e., the calculations are made at about the upper limit of applicability of the analogy). Additionally, the aspect ratio relation quoted [Eq. (1)] was obtained from measured data for post-buckled configurations with a considerably smaller number of facets.

Even allowing for errors associated with the aspect ratio and the effective width of flange it is apparent that for very thin near perfect cylinders a substantial reduction in the buckling load from the classical load is a certainty. The reduction appears to be not just due to initial imperfections but rather a deformation mode developed during loading.

References

[1]Hart-Smith, L. J., "Buckling of Thin Cylindrical Shells under Uniform Axial Compression," *International Journal of Mechanical Science,* Vol. 12, April 1970, pp. 299-313.

[2]Babcock, C. D. and Sechler, E. E., "The Effect of Initial Imperfections on the Buckling Stress of Cylindrical Shells," NASA TN D1510, Dec. 1962, pp. 135-142.

[3]Foster, C. G., "Some Observations on the Yoshimura Buckle Pattern for Thin-Walled Cylinders," *Transactions of the ASME Journal of Applied Mechanics,* Vol. 46, June 1979, pp. 377-380.

[4]Foster, C. G., "Estimation of the Collapse Loads of Thin-Walled Cylinders in Axial Compression," *Transactions of the ASME Journal of Applied Mechanics,* Vol. 46, June 1979, pp. 381-385.

[5]Foster, C. G., "On the Buckling of Thin-Walled Cylinders Loaded in Axial Compression," *Journal of Strain Analysis for Engineering Design,* Vol. 16, July 1981, pp. 205-210.

[6]Yoshimura, Y., "On the Mechanism of Buckling of a Circular Cylindrical Shell under Axial Compression," NACA TM 1390, 1955.

[7]Tennyson, R. C., "Buckling Modes of Circular Cylindrical Shells under Axial Compression," *AIAA Journal,* Vol. 7, Aug. 1969, pp. 1481-1487.

Orthogonal Polynomials as Variable-Order Finite Element Shape Functions

Dewey H. Hodges*
U.S. Army Research and Technology Laboratories (AVRADCOM), Ames Research Center, Moffett Field, California

Introduction

VARIABLE-order finite element methods have an advantage over conventional finite element methods in that additional degrees of freedom may be added to the model without the need to generate a new element geometry. These additional element degrees of freedom generally are not physical displacements or rotations of some point at the element boundary. Instead, they are typically generalized coordinates associated with a set of polynomial shape functions of a variable order higher than that of the conventional polynominal shape functions used in defining the displacement field of the element in terms of displacements and rotations at the boundary. In some cases the variable-order polynomials may be organized so that the corresponding generalized coordinates may be displacements, rotations, or derivatives of displacement of some set of points within the element.

Previous papers[1,2] indicate that, when elements are chosen to be free of discontinuities in properties, for a given number of degrees of freedom in the total structure, higher accuracy is obtained using fewer elements with polynomials of higher order. For the range of problems investigated in Ref. 1, there are also corresponding improvements in computational effort. Results from the author's unpublished work in both static and dynamic, linear and nonlinear, and conservative and nonconservative problems indicate that the same trends hold for a wide class of one-dimensional problems.

Although the shape functions in Ref. 1 lead to well-conditioned element matrices for beams, the shape functions presented therein lack certain computational advantages.

1) The element matrices that result from the shape functions are not sparse.

2) The element matrices are not hierarchical; that is, matrices that result from shape functions of a given order are not submatrices of the ones obtained from shape functions of a higher order. This is because all the shape functions change each time the order is increased.

3) Since the variable-order shape functions contain displacements and rotations at the boundary, the corresponding generalized coordinates cannot be eliminated from the global equations in a linear static problem. The purpose of this Note is to present variable-order polynomial shape functions that overcome these disadvantages.

In typical applications of finite element methods, the unknown functions are expanded in a series of polynomials in each element and certain relationships are enforced at the element boundaries. For one-dimensional structures, the relationships may be either between values of the functions themselves or between values of the functions and first derivatives of the functions. The shape functions for the former are typically referred to as C0 type shape functions while the latter are C1 type. In this Note variable-order shape functions of both types are presented.

Received May 17, 1982; revision received July 20, 1982.

*Research Scientists, Rotorcraft Dynamics Division, Aeromechanics Laboratory. Associate Fellow AIAA.

C0 Type Shape Functions

The most common C0 type shape functions for one-dimensional structures are the linear functions

$$S_1 = 1 - x \qquad S_2 = x$$

for the interval [0,1]. It is desired that the higher order functions vanish at the end points $x=0$ and $x=1$ so that the function to be expanded has the value at $x=0$ of the coefficient of S_1 and the value at $x=1$ of the coefficient of S_2. A set of orthogonal, hierarchical functions that satisfies these conditions is given by

$$S_{n+3} = x(1-x)\,G_n(5,3,x) \qquad (n=0,1,\ldots)$$

These functions vanish at the end points and are hierarchical in the sense described earlier. Here, $G_n(5,3,x)$ are Jacobi polynomials[3] which are orthogonal over the interval [0,1] with the weighting function $x^2(1-x)^2$. Thus, C0 type shape functions S_i are orthogonal to S_j when i and j are greater than 2 and not equal to each other. Since $G_n(5,3,x)$ is a polynomial of degree n, the C0 function S_{n+3} is of degree $n+2$ for $n=0,1,\ldots$. In Ref. 3 the recurrence relations that are needed to generate Jacobi polynomials are given.

C1 Type Shape Functions

The most common C1 type shape functions for one-dimensional structures are the cubic functions

$$S_1 = 1 - 3x^2 + 2x^3 \qquad S_2 = x - 2x^2 + x^3$$

$$S_3 = 3x^2 - 2x^3 \qquad S_4 = -x^2 + x^3$$

for the interval [0,1]. It is desired that the higher order functions and their first derivatives vanish at the end points $x=0$ and $x=1$. This is so that the function to be expanded and its first derivative have the values at $x=0$ of the coefficients of S_1 and S_2, respectively. Similarly, the function and its first derivative have the values at $x=1$ of the coefficients of S_3 and S_4, respectively. A set of orthogonal, hierarchical functions that satisfies these conditions is given by

$$S_{n+5} = x^2(1-x)^2 G_n(9,5,x) \qquad (n=0,1,\ldots)$$

These functions and their first derivatives vanish at the end points and are hierarchical in the sense described earlier. Here, $G_n(9,5,x)$ are Jacobi polynomials[3] which are orthogonal over the interval [0,1] with the weighting function $x^4(1-x)^4$. Thus, C1 type shape functions S_i are orthogonal to S_j when i and j are greater than 4 and not equal to each other. Since $G_n(9,5,x)$ is a polynomial of degree n, the C1 function S_{n+5} is of degree $n+4$ for $n=0,1,\ldots$.

Discussion

Linear dynamic results obtained from the present shape functions are identical to those obtained in Ref. 4 for the C0 case and identical to those obtained in Refs. 1,5, and 6 for the C1 case. In Refs. 4-6 the element displacement is expanded in a simple power series for which conditioning problems are known to occur. The element matrices obtained from the present shape functions, however, are so well conditioned, that polynomials of order 15 and higher have been used in the identical manner as in Ref. 1 with virtually no sign of ill conditioning on a 60-bit-word computer with single precision (14-place) arithmetic. In addition, the present shape functions lead to sparse, hierarchical matrices. The orthogonality of the higher order shape functions contributes to the sparsity and overall good conditioning of these matrices.

The higher order shape functions presented herein are a special case of general higher order shape functions in that they (and their first derivatives in the case of C1 type functions) vanish at the element boundaries. This implies that their corresponding generalized coordinates can be eliminated from the global equations in static problems. This fact can be used to significant advantage, particularly in nonlinear problems, in that equations for the generalized coordinates associated with each element can be solved at the element level, generally leading to a savings in storage and computer operations.

Acknowledgments

Technical discussions with David C. Galant of the NASA Ames Research Center and with Dr. Michael J. Rutkowski of the Aeromechanics Laboratory are gratefully acknowledged.

References

[1] Hodges, D.H. and Rutkowski, M.J., "Free Vibration Analysis of Rotating Beams by a Variable-Order Finite-Element Method," *AIAA Journal*, Vol. 9, Nov. 1981, pp. 1459-1466.

[2] Szabo, B.A., "Some Recent Developments in Finite Element Analysis," *Computers and Mathematics with Applications*, Vol. 5, 1979, pp. 99-115.

[3] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1970, Chap. 22.

[4] Hodges, D.H., "Direct Solutions for Sturm-Liouville Systems with Discontinuous Coefficients," *AIAA Journal*, Vol. 17, Aug. 1979, pp. 924-926.

[5] Bailey, C.D., "Exact and Direct Analytical Solutions to Vibrating Systems with Discontinuities," *Journal of Sound and Vibration*, Vol. 44, No. 1, 1976, pp. 15-25.

[6] Hodges, D.H., "Vibration and Response of Nonuniform Rotating Beams with Discontinuities," *Journal of the American Helicopter Society*, Vol. 24, Oct. 1979, pp. 43-50.

Extrapolation of Optimum Design Based on Sensitivity Derivatives

Jean-Francois M. Barthelemy*
Virginia Polytechnic Institute and State University, Blacksburg, Virginia
and
Jaroslaw Sobieszczanski-Sobieski†
NASA Langley Research Center, Hampton, Virginia

Introduction

In many applications, it is useful to know how the perturbations of the design problem constants (parameters) will affect the problem optimum solution. For this purpose, optimum sensitivity analysis has been proposed in Ref. 1 as a structural design tool to assess the parameter perturbation effects by extrapolations based on optimum sensitivity derivatives, without resorting to relatively costly reoptimization. With respect to the extrapolation accuracy, the numerical experience in structural applications reported to date[1,2] showed that, in most cases, accuracy sufficient for engineering purposes is attainable over a range in excess of 20% of the parameter value.

The purpose of this Note is to contribute to the aforementioned experience a test case deliberately chosen so as to have a strong nonlinearity of the constraints, with

Received Jan. 4, 1982; revision received July 19, 1982.

*Research Associate, Department of Aerospace and Ocean Engineering. Student Member AIAA.

†Head, Multidisciplinary Analysis and Optimization Branch, Loads and Aeroelasticity Division. Associate Fellow AIAA.